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Thin slender water jets

By M. S. BORGAS AND E. O. TUCK
Department of Applied Mathematics, University of Adelaide, S. Australia 5001

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An analysis is provided for the free development of slender jets of water, in which the cross-sections are of small thickness-to-width ratio.

1. Introduction

If a jet of water emerges at sufficient speed from a nozzle, the flow-velocity vector will be directed almost along the jet, and its cross-section will change only slowly. Such jets, e.g. from garden hoses, are readily observable, and have been described by many authors, e.g. Rayleigh (1945, p. 355). The problem of theoretical determination of the shape of the cross-section of the jet is not, however, an easy one and only recently have such analyses been performed.

Taylor (1960) and Longuet-Higgins (1972) neglected gravity and discussed the case when the sections are initially (and remain) elliptical. Tuck (1976) incorporated gravity into the elliptic-section case; remarkably, elliptic sections can remain elliptic, even when perturbed laterally by gravity. Geer & Strikwerda (1980) carried out direct numerical computations on jets of general section. Substantial and progressively increasing distortion occurs for non-elliptic initial cross-sections.

In the present paper, we analyse a special limiting case in which analytic predictions of this distortion are possible, namely that in which individual cross-sections have a small thickness-to-width ratio. For simplicity, we neglect gravity, and hence may assume that the main jet is straight, lying close to the x-axis, and thinner in the z-direction than the y-direction.

The results determine the evolution of the jet in terms of two functions $F(y)$ and $V(y)$, the former being such that $z = F(y)$ is the shape of the jet at an initial section $x = 0$, while the latter function $V(y)$ measures the (lateral) component of fluid velocity at that initial section. Procedures are outlined for determining $F(y)$ and $V(y)$ from the geometrical properties of the nozzle.

The most important determinant of the jet's cross-section is the lateral velocity $V(y)$. If $V(y)$ is zero or constant in value, the jet retains its original shape for ever, in the absence of gravity. If $V(y)$ varies linearly with y, the jet expands or contracts in a self-similar manner. If it is contracting, it must reach a point of zero width at some subsequent value of x. The present approximate theory predicts a singularity at this x, where the thickness becomes infinite. However, comparison with the exact theory for elliptical jets that have linear $V(y)$ indicates that this singularity simply models a rapid switch of the major and minor axes of the jet cross-section.

For general $V(y)$, singularities may develop at sections x of non-zero width, and examples for a quadratic function $V(y)$ are shown here, with singularities either at the edge or at the centre plane of symmetry. These correspond to the jet suddenly 'sprouting' side sheets, as is known to happen in some cases.

2. Derivation of equations for free jet

The water is assumed to be inviscid and incompressible, and to be moving steadily and irrotationally. Its velocity potential $\phi(x,y,z)$ therefore satisfies Laplace's equation

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{2.1}$$

in the region of flow. That region is supposed to have boundaries

$$z=\pm f(x,y), \quad |y|<b(x). \tag{2.2}$$

That is, the jet has width $2b(x)$, measured in the y-direction, and thickness $2f(x,y)$ in the z-direction. We concentrate on the boundary $z=+f$, invoking symmetry with respect to the plane $z=0$.

The kinematic boundary condition is

$$\phi_z=f_x\phi_x+f_y\phi_y \quad \text{on} \quad z=f, \tag{2.3}$$

and, in the absence of gravity and surface tension, constancy of pressure at the jet boundary implies constancy of velocity magnitude, i.e.

$$\phi_x^2+\phi_y^2+\phi_z^2=U^2 \quad \text{on} \quad z=f, \tag{2.4}$$

for some constant U. The most general free-jet problem is to solve (2.1), subject to (2.3) and (2.4) on the unknown surface $z=f$.

The slender-jet assumption is that the flow is predominantly in the x-direction. In the absence of gravity or compressibility, that dominant flow velocity must be independent of x and of magnitude U, and hence we may write

$$\phi=Ux+\Phi, \tag{2.5}$$

for some small perturbation potential Φ. The equation satisfied by Φ is then

$$\Phi_{yy}+\Phi_{zz}=0, \tag{2.6}$$

subject to

$$\Phi_z=Uf_x+f_y\Phi_y, \tag{2.7}$$

$$2U\Phi_x+\Phi_y^2+\Phi_z^2=0 \tag{2.8}$$

on $z=f$. That is, as in aerodynamic slender-body theory (Thwaites 1960) the cross-flow potential Φ satisfies the two-dimensional Laplace equation in planes $x=$ const. The boundary $z=f$ of this cross-section is still unknown, and is determined by the coupled nonlinear boundary conditions (2.7), (2.8).

The system (2.6)–(2.8) is solvable by numerical means, given any initial configuration $f(0,y)$, $\Phi(0,y,z)$. Some such solutions, with gravity also included, were developed by Geer & Strikwerda (1980).

A further simplification is obtained if the jet is not only slender, but also thin, namely if $f \ll b$. Then Φ can be replaced by its Taylor series about $z=0$, namely

$$\Phi(x,y,z)=\Phi(x,y,0)-\tfrac{1}{2}z^2\Phi_{yy}(x,y,0)+\dots. \tag{2.9}$$

Now the boundary conditions (2.7), (2.8) reduce respectively to

$$Uf_x + (f\Phi_y)_y = 0, \tag{2.10}$$

$$2U\Phi_x + \Phi_y^2 = 0, \tag{2.11}$$

with Φ written for $\Phi(x, y, 0)$. It is convenient to introduce the lateral velocity component

$$v(x,y) = \Phi_y(x,y,0),$$

in terms of which (2.10) gives

$$Uf_x + (fv)_y = 0, \tag{2.12}$$

and (2.11) gives, upon differentiation with respect to y,

$$Uv_x + vv_y = 0. \tag{2.13}$$

Equations (2.12), (2.13) are the fundamental equations for a thin slender jet. Equation (2.13) determines the velocity $v(x,y)$, and may be thought of as an approximation to the Euler equation of hydrodynamics, for conservation of y-wise momentum, in the absence of pressure gradient. Once v is determined, the continuity equation (2.12) determines the cross-section shape $f(x,y)$. Equations (2.12), (2.13) take an especially familiar appearance if we note that the co-ordinate x plays only a time-like role, and introduce a pseudo-time co-ordinate $t = x/U$. However, we shall retain the original x-co-ordinate here.

3. Solution for free jets

The first-order partial differential equations (2.12), (2.13) can be solved completely by the method of characteristics (cf. Ames 1965, p. 51). We need merely quote the final result, expressed in implicit form, namely

$$v(x,y) = V(Y), \tag{3.1}$$

$$f(x,y) = \frac{F(Y)}{1 + xV'(Y)/U}, \tag{3.2}$$

where

$$Y = y - \frac{vx}{U}. \tag{3.3}$$

Although the functions $V(Y)$, $F(Y)$ occurring in (3.1), (3.2) are in general arbitrary, they can be interpreted as 'initial' values of v and f respectively, since it follows that

$$v(0,y) = V(y), \tag{3.4}$$

$$f(0,y) = F(y). \tag{3.5}$$

That is, (3.1)–(3.3) enable complete determination of the shape and lateral velocity of the jet for all $x > 0$, given initial values of these quantities at $x = 0$.

In particular, if $v(0,y) \equiv 0$, then $V \equiv 0$, and hence $v(x,y) \equiv 0$ for all $x > 0$. Also $f(x,y) = f(0,y)$ for all $x > 0$. That is, if at the initial section $x = 0$ there is no lateral velocity component, then no such component ever develops, and the jet retains the original cross-section for ever. Our interest is therefore mainly with initial configurations that possess a non-zero lateral flow v.

The simples such possibility (aside from v = const. which is of no interest) is a linear expression, i.e.

$$V(y) = ky, \tag{3.6}$$

for some constant k. Now the implicit equation (3.1) subject to (3.3) can be solved to give the explicit result

$$v(x,y) = \frac{ky}{1+kx/U}, \tag{3.7}$$

from which follows

$$f(x.y) = \frac{F(y/(1+kx/U))}{1+kx/U}. \tag{3.8}$$

That is, in this case, the jet retains its initial cross-section shape, but is stretched by the factor $1+kx/U$ in the y-direction, and correspondingly shrunk by the inverse of this same factor in the z-direction. The combined effect of these two stretchings conserves mass, as required. The factor $1+kx/U$ exceeds unity if $k > 0$, i.e. if at the initial cross-section the fluid is expanding, and in such cases the jet continues to expand laterally, while becoming ever thinner. Conversely, if $k < 0$, the jet contracts laterally and grows in thickness. Eventually, this growth in thickness must invalidate the basic thin-jet assumption, and does so prior to the collapse of the edges onto each other at $x = -U/k$.

The above example applies irrespective of the initial cross-section shape $F(y)$, providing the initial velocity V is linear in y. It is in principle possible to design a nozzle of a suitable converging or diverging character to achieve this result for any $F(y)$. However, a particular case of interest is an elliptical shape, i.e.

$$F(y) = c_0(1-y^2/b_0^2)^{\frac{1}{2}}. \tag{3.9}$$

If the ellipse (3.9) is attained by smooth variation of the form

$$f(x,y) = c(x)\,(1-y^2/b(x)^2)^{\frac{1}{2}} \tag{3.10}$$

for $x \leqslant 0$, with $c_0 = c(0\_)$, $b_0 = b(0\_)$, i.e. in a slender nozzle whose every section is an ellipse of semi-axes $(b(x), c(x))$, then the lateral velocity is indeed linear in y, and has

$$k = \frac{U}{b_0}b'(0\_). \tag{3.11}$$

Thus, as might be expected, the jet expands or contracts according to whether or not the nozzle from which it emerges is expanding or contracting, and is dependent on the sign of $b'(0\_)$. In fact, these results are simply the limit as $c_0/b_0 \to 0$ of the exact results of Taylor (1960) for elliptical jets.

If, in general, we write the lateral edge of the jet as $y = b(x)$, $x > 0$, then this must be a continuation, via the implicit relationships (3.1)–(3.3), of the edge $y = b_0$ of the initial cross-section. That is, suppose that $F(y)$ is a function that vanishes only at $y = \pm b_0$. Then $y = b(x)$ where $F(Y) = 0$, i.e.

$$b(x) = b_0 + \frac{V(b_0)\,x}{U}. \tag{3.12}$$

That is, the jet edge is always a straight line, expanding or contracting linearly with distance from the nozzle according to the sign of the lateral velocity component at the edge of that nozzle. Even for non-elliptic sections it may be shown that

$$V(b_0) = Ub'(0\_), \tag{3.13}$$

where $b'(0\_)$ is the longitudinal slope of the lateral edge of the nozzle at its mouth. That is, the jet plan form is $y = \pm b(x)$, where

$$b(x) = b_0 + xb'(0\_). \tag{3.14}$$

The factor $1 + xV'(Y)/U$ in the denominator of (3.2) is an amplification factor for the lateral extent of the jet, and its inverse is a thickness growth factor, in the general case. However, because of the implicit involvement of v, the jet does not simply stretch itself without distortion, unless V is a linear function as in (3.6). Nevertheless, a general conclusion can be reached in cases where $V'(y)$ takes significant *negative* values. That is, in such cases, for sufficiently large x, the factor in the denominator of (3.2) may vanish, and hence a singularity develops in the jet-thickness distribution.

This singularity is mathematically equivalent to that for a shock wave in gas-dynamics (cf. Shapiro 1953, chap. 8) or a hydraulic jump in hydraulics (cf. Streeter 1961, chap. 3). Indeed there is a close connection between the governing equations (2.12), (2.13) and equations describing one-dimensional flows in these fields. However, the physical interpretation of the singularity is not necessarily that of a shock in the present application.

For example, we have already seen in the special case of a linear velocity profile (3.6) that this singularity appears exactly at the value of x where the jet's width has reduced to zero. That is, all that is happening in this case is that the jet is switching over from its initial thin sheet form, parallel to the y-axis, to a corresponding thin sheet parallel to the z-axis. In fact, it is known (Taylor 1960) that an elliptic free jet must always go through such a transition, and the 'shock' at $x = -U/k$ is no more than an indication of that transition taking place.

4. Quadratic lateral velocity profiles

Further insight into the behaviour of thin jets is obtained by use of the simple quadratic profile

$$V(y) = ky + \eta y^2 \quad (y > 0), \tag{4.1}$$

which generalizes the linear profile (3.6). For any $\eta \neq 0$, the jet cross-section must distort, and our interest is in the nature of this distortion, and, in particular, in the development of singularities.

In fact, because a scaling can always be performed of the type $x \to kx$, $V \to k^{-1}V$, without changing the problem, we can confine our attention to three one-parameter families: (i) $k = 0$; (ii) $k = 1$; (iii) $k = -1$. That is, an overall doubling of the lateral velocity at the initial cross-section simply contracts the jet in the x-direction, and the distortion occurs twice as fast. For definiteness, we also normalize the jet's initial half-width to unity, writing $b_0 = 1$, and hence confining attention to $0 \leqslant Y \leqslant 1$, and set $U = 1$.

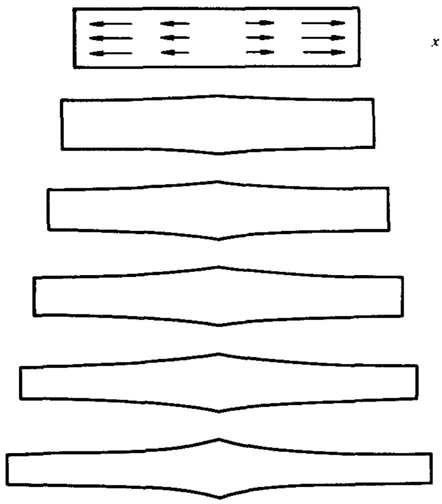

FIGURE 1. Evolution of a free jet from a rectangular aperture at which the lateral velocity distribution is $V(y) = y^2$.

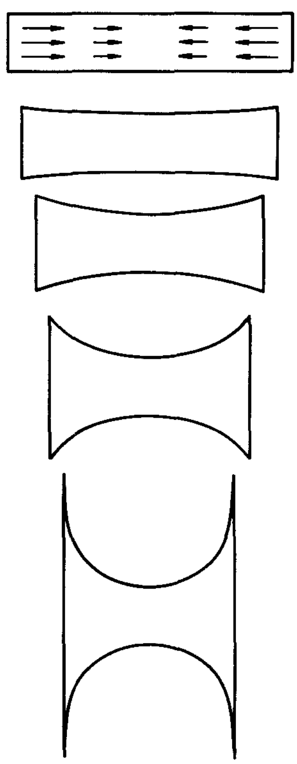

FIGURE 2. As figure 1, with $V(y) = -y^2$.

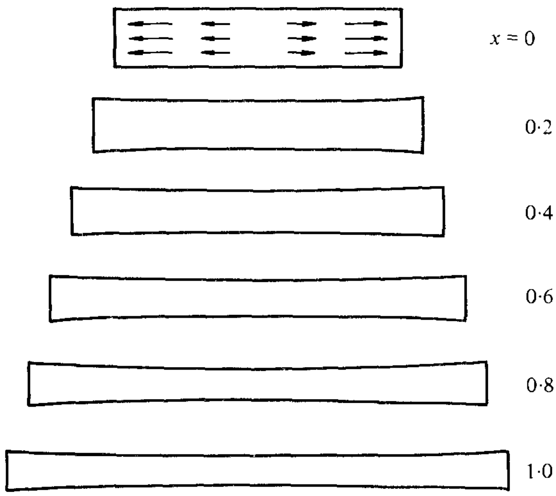

FIGURE 3. As figure 1, with $V(y) = y - 0{\cdot}25y^2$.

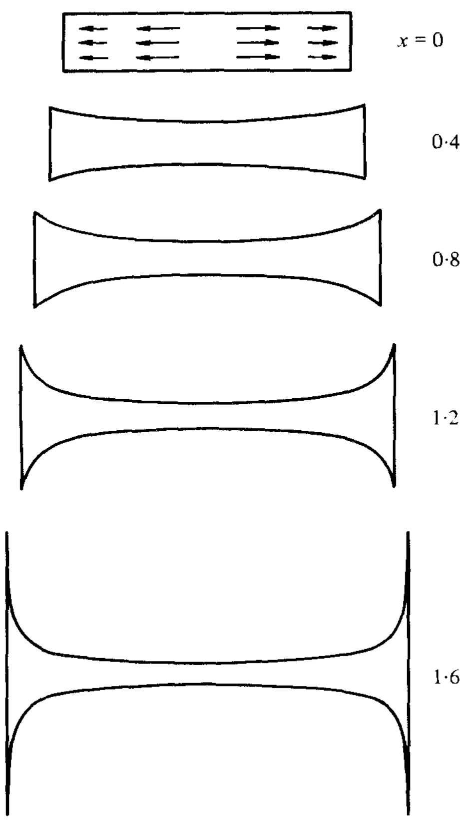

FIGURE 4. As figure 1, with $V(y) = y - 0{\cdot}75y^2$.

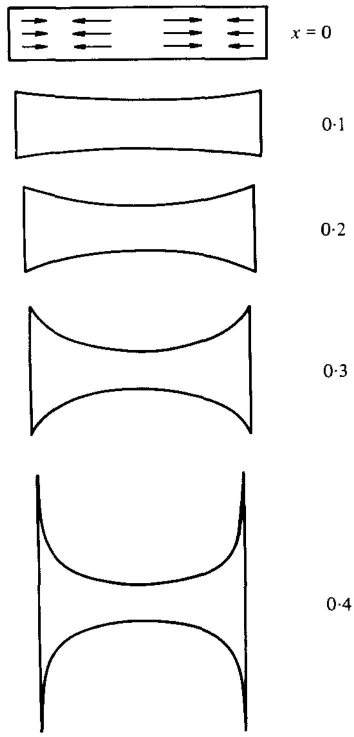

FIGURE 5. As figure 1, with $V(y) = y - 1{\cdot}5y^2$.

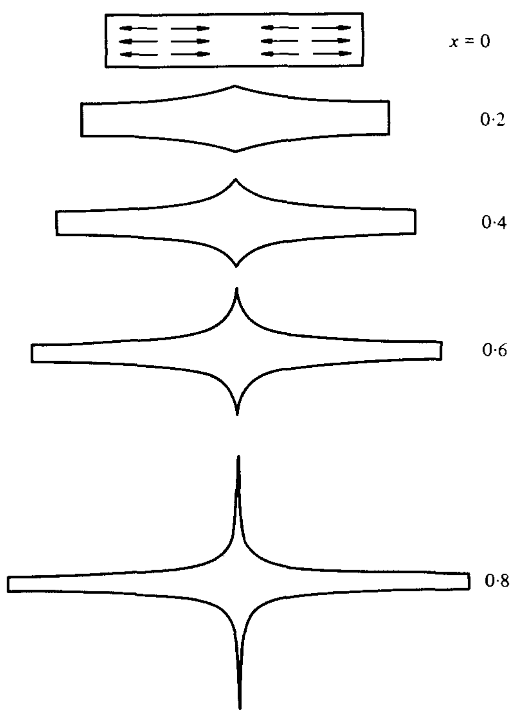

FIGURE 6. As figure 1, with $V(y) = -y + 2y^2$.

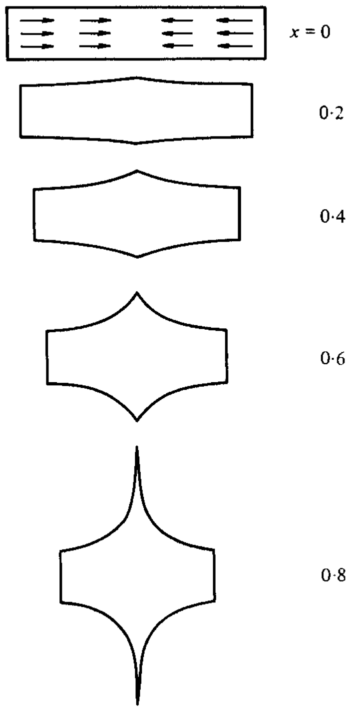

FIGURE 7. As figure 1, with $V(y) = -y+0{\cdot}5y^2$.

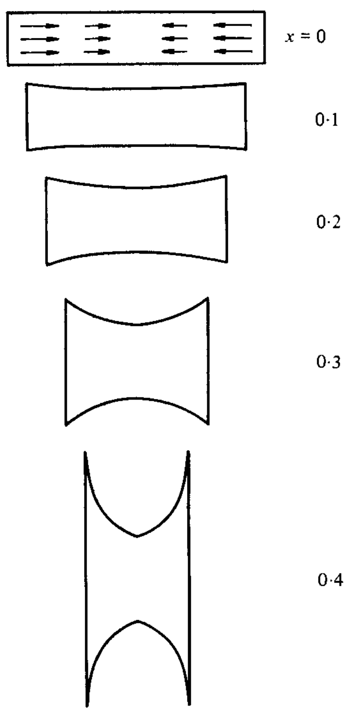

FIGURE 8. As figure 1, with $V(y) = -y-0{\cdot}5y^2$.

A primary quantity of interest is the section $x = x_s$ at which, for the first time, the denominator of (3.2) changes sign from positive to negative. Clearly the thin-jet assumption has broken down at (in practice just before) this section x_s. Such a singularity can never occur if $V'(Y) > 0$. That is, if the jet is initially expanding everywhere, it continues to do so in a perfectly stable manner, and becomes ever thinner and wider as x increases. This is true in family (i) if $\eta > 0$ and family (ii) if $\eta > -\frac{1}{2}$, and hence we have no further interest in these cases.

Since the expression in the denominator of (3.2) is a linear function of Y, its minimum value in $0 \leqslant Y \leqslant 1$ necessarily occurs either at $Y = 0$ or $Y = 1$. That is, the singularity first arises either in the middle or at the edge of the jet. In fact the parameter η determines this question, the singularity occurring at the edge if $\eta < 0$ and at the centre if $\eta > 0$.

For the family (i) with $\eta < 0$, the singularity must occur first at the edge $Y = 1$, i.e. we have

$$x_s = -\frac{1}{2\eta},$$

and the corresponding half-width of the jet is

$$y_s = \tfrac{1}{2}$$

for all η. That is, this family always contracts to exactly half its original width, at which point it develops a singularity at the edge.

Family (ii) is only of interest if $\eta < -\frac{1}{2}$, and the singularity again always occurs first at the edge $Y = 1$, at the section

$$x_s = -\frac{1}{2\eta+1},$$

with the jet having a half-width

$$y_s = \frac{\eta}{2\eta+1}.$$

It is of interest to note that, for $-1 < \eta < -\frac{1}{2}$, the jet is expanding laterally. Cases with $\eta < -1$, on the other hand, are like family (i) in that the jet is always contracting laterally; in fact, as $\eta \rightarrow -\infty$, this family must coincide with family (i).

Family (iii) can give a singularity both for $\eta > 0$ and $\eta < 0$. In fact it is never stable, for any η. If $\eta < 0$, the initial velocity is everywhere inward as for family (i), the jet converges laterally, and the singularity first develops at the edge, at

$$x_s = \frac{1}{1-2\eta},$$

where the half-width is

$$y_s = -\frac{\eta}{1-2\eta}.$$

If $\eta > 0$, the singularity first develops at the centre $t = 0$, and hence must occur at the same section $x_s = 1$ for all η. The half-width at this point is $y_s = \eta$. Note that this jet is contracting if $0 < \eta < 1$, and expanding if $\eta > 1$.

The linear-profile results discussed in § 3 are recovered from this family (iii) as $\eta \to 0$. In particular, $y_S \to 0$ as $\eta \to 0$. That is, only for $\eta = 0$ does the singularity correspond to a direct change between a sheet that is then with respect to z and one that is thin with respect to y; in all other cases the new jet has at least two 'planes of thinness'.

Figures 1–8 show samples of the cross-sections that develop from an initially rectangular jet in which the lateral velocity profile is quadratic. That is

$$F(Y) = \epsilon = \text{const.} \quad (0 < Y < 1).$$

For definiteness, ϵ is shown as 0·2, although this thickness scale is of course quite arbitrary. The family (i) cases of figures 1 and 2 are the stable case $\eta = 1$ and the case $\eta = -1$ where the edge singularity occurs at $x = 0{\cdot}5$.

Similarly, figures 3–5 are for family (ii), and include respectively a stable case $\eta = -0{\cdot}25$, an expanding case $\eta = -0{\cdot}75$ with an edge singularity at $x = 2$, and a contracting case $\eta = -1{\cdot}5$ with an edge singularity at $x = 0{\cdot}5$. Finally, figures 6–8 are for family (iii), and include respectively an expanding case $\eta = 2$ with a centre singularity at $x = 1$, a contracting case $\eta = 0{\cdot}5$ also with centre singularity at $x = 1$, and a contracting case $\eta = -0{\cdot}5$ with an edge singularity at $x = 0{\cdot}5$.

5. Thin slender nozzles

The free jets discussed above must have been produced by a nozzle of some kind, and the parameters $F(Y)$, $V(Y)$ at the initial section $x = 0$ are determined by matching with such a nozzle. A particular case of interest is that when, like the jet itself, the nozzle is both slender and thin. That is, the jet can be considered as a continuation into $x > 0$ of a similarly bounded flow in $x < 0$.

Thus, equations describing the flow in the nozzle region $x < 0$ can be obtained by a similar derivation to that in § 2. Clearly the constant-pressure boundary condition (2.4) no longer applies, and hence we delete the resulting momentum equation (2.13). The flow is described by one equation only, a continuity equation like (2.12); fortunately there is now only one unknown function $v(x, y)$, since the nozzle geometry prescribes the thickness function $f(x, y)$.

However, the continuity equation is not necessarily identical to (2.12), since, in the absence of the constant-pressure condition (2.4), there is no reason to retain the assumption that the dominant x-wise velocity component U is independent of x. That is, if the nozzle is contracting or expanding (in cross-sectional area) the mean flow may (indeed *must*) vary along the jet, with $U = U(x)$.

The corresponding corrections to the derivation in § 2 lead in a straightforward manner to a modified continuity equation, namely

$$(Uf)_x + (fv)_y = 0, \tag{5.1}$$

which reduces to (2.12) if U is constant. Although (5.1) appears to have introduced another unknown $U(x)$, integration of (5.1) with respect to y across the section $x =$ const. provides the Venturi law

$$U(x)\,S(x) = \text{const.}, \tag{5.2}$$

where

$$S(x) = \int f(x, y)\,dy \tag{5.3}$$

is the nozzle's cross-section area. For a general nozzle shape, prescribed by a given function $f(x,y)$, we can determine the lateral velocity profile $v(x,y)$ simply by integrating (5.1), having computed $U(x)$ by (5.2).

An interesting class of nozzles is the self-similar family

$$f(x,y) = c(x)\,\Omega\left(\frac{y}{b(x)}\right), \tag{5.4}$$

where $\Omega = \Omega(t)$ $(-1 \leqslant t \leqslant 1)$ is a given function. That is, this nozzle's sections are all of the same shape, but stretched in an arbitrary manner, by $b(x)$ in the y-direction and independently by $c(x)$ in the z-direction. We see immediately that

$$S(x) = S_0\, b(x)\, c(x), \tag{5.5}$$

where

$$S_0 = 4\int_0^1 \Omega(t)\,dt, \tag{5.6}$$

and hence

$$U(x) = \frac{U_0 b_0 c_0}{b(x)\,c(x)} \tag{5.7}$$

if $U = U_0$ when $b = b_0$, $c = c_0$. Now (5.1) states that

$$(\Omega v)_y = U_0 b_0 c_0 \frac{b'(x)}{b(x)^2 c(x)}\left[\Omega + \frac{y}{b(x)}\Omega'\right],$$

which integrates to give

$$v(x,y) = U_0 b_0 c_0 \frac{b'(x)}{b(x)^2 c(x)}\, y. \tag{5.8}$$

That is, in all members of this self-similar family, the lateral-velocity profile in the nozzle is everywhere proportional to y. Hence, in particular, the profile at the aperture $x = 0$ is linear in y, and (3.6) holds, with (3.11) determining the constant k.This family includes that of elliptic sections discussed in §3, and also all rectangular-sectioned nozzles, and many other important cases. The results of §3 show then that, in all such families, if $k < 0$, the jet develops a 'singularity' at its centre $y = 0$, at a point $x = x_s$ where the jet width $b(x_s)$ has reduced to zero. All such jets are simply switching over suddenly, from thin in the y-direction, to thin in the z-direction.

The quadratic velocity profiles discussed in §4 can be achieved in many types of thin slender nozzles. For example, suppose the nozzle has a trapezoidal shape for $y > 0$, i.e. $f(x,y)$ is a general linear function of y, of the form

$$f(x,y) = c(x)\,[1 + \mu(x)\,y] \quad (0 < y < b(x)). \tag{5.9}$$

If $\mu = 0$, the section is a rectangle of dimensions $b(x)$ by $c(x)$. Now, for general $\mu(x)$, the cross-section area is

$$S(x) = 4c(b + \tfrac{1}{2}\mu b^2), \tag{5.10}$$

and hence the fluid velocity is

$$U(x) = U_0 \frac{c_0(b_0 + \frac{1}{2}\mu_0 b_0^2)}{c(b + \frac{1}{2}\mu b^2)}. \tag{5.11}$$

Thus (5.1) states that

$$(Uc)' + (Uc\mu)'y + (fv)_y = 0,$$

which integrates to give

$$v(x,y) = -\frac{y}{c}\left[\frac{(Uc)' + \frac{1}{2}(Uc\mu)'y}{1 + \mu y}\right]. \tag{5.12}$$

The profile (5.12) is parabolic at any *rectangular* section x, where $\mu = 0$. If we take $\mu = 0$ at the aperture $x = 0$, and evaluate the derivatives at that point, we find that, as in (4.1),

$$V(y) = v(0,y) = ky + \eta y^2,$$

where

$$k = \frac{U_0}{b_0}[b_0' + \tfrac{1}{2}\mu_0' b_0^2], \tag{5.13}$$

$$\eta = -\tfrac{1}{2}U_0\mu_0'. \tag{5.14}$$

Thus any departure ($\eta \neq 0$) from a linear profile demands that $\mu_0' = \mu'(0) \neq 0$, i.e. that the sections immediately ahead of the rectangular aperture be non-rectangular. Equation (5.13) is a direct modification of the equation (3.11) that holds for linear profiles.

A similar analysis can be performed for many classes of slender nozzle geometry. Thin slender jets can also be created by non-slender nozzles. For example, we could create a *sudden* contraction forming a thin aperture at the exit from an otherwise-circular hose-pipe. In such a case there will be a matching region where the flow direction is not necessarily nearly parallel to the x-direction, before the free thin slender jet is established. Further work is needed to determine the functions $F(y)$ and $V(y)$ for such a free jet, from the geometrical properties of the aperture.

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